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CONTROL OF AN ELASTIC MANIPULATOR ARM USING LOAD POSITION AND VELOCITY FEEDBACK†

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The rotation of an elastic manipulator arm about one of its ends in the horizontal plane is investigated. A load is attached to the other end. The motion is effected by an electric motor. The control is constructed in the form of linear feedback on the position of the load, its velocity, and the angular velocity of the arm. The stability of the control process is investigated. It is shown that when there are no viscous damping forces proportional to the angular velocity of the arm, load position and velocity feedback leads to undamped oscillations of the system and the desired equilibrium position is not stabilized. Asymptotic stability domains in the feedback coefficient space when viscous damping is present are constructed. Comparison shows these domains to be smaller than corresponding domains for a completely rigid body.

1. THE EQUATIONS OF MOTION

CONSIDER an elastic homogeneous rod of length l and constant transverse cross-section. The rod can rotate in the horizontal plane (the plane of the diagram) about one of its ends: the fixed point O (Fig. 1). Attached to the other end of this arm is the object to be manipulated: a load, which we shall assume to be a point of mass M. Figure 1 shows the neutral line OM of the (bent) rod, which always lies in the horizontal plane. The moving coordinate axis OX touches the neutral line at the point O. the angle between the OX axis and some fixed direction OP is denoted by α. We denote by $\upsilon(x,t)$ the deviation at a time t of the point with coordinate x on the neutral line OM from the OX axis. The rotation of the arm is controlled by an electric motor (not shown in Fig. 1) through a reduction gear with reduction coefficient k. Let J be the moment of inertia of the motor armature and φ the angle of rotation of the armature, so that $\dot\varphi(t)=k\dot\alpha(t)$.

In the linear theory of thin straight inextensible rods [1, 2] the equations of motion of this mechanical system can be written in the form [3–5]

$$EI\upsilon''''(x,t)+\rho(\ddot\upsilon(x,t)+x\ddot\alpha(t))=0 \tag{1.1}$$

$$J\ddot\varphi(t)=Q+(EI/k)\upsilon''(0,t) \tag{1.2}$$

$$\upsilon(0,t)=\upsilon'(0,t)=\upsilon''(l,t)=0$$

$$M(\ddot\upsilon(l,t)+l\ddot\alpha(t))=EI\upsilon'''(l,t) \tag{1.3}$$

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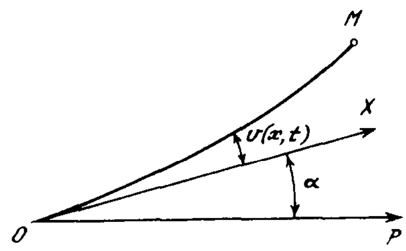

FIG. 1.

Here ρ is the density per unit length of the material, E is Young's modulus, I is the constant moment of inertia of a transverse section of the rod about the vertical axis, and Q is the electromagnetic torque about the axis of the armature.

Relation (1.1) describes plane transverse oscillations of the rod [1, 2, 6] for a specified angular acceleration $\ddot{\alpha}$. It ignores energy dissipation during oscillations. Relation (1.2) describes the variation of the angular momentum of the motor armature; the second term on its right-hand side describes the torque imposed on the armature by the elastic rod; the inertia of the reduction gears is ignored. Equations (1.1)–(1.3) omit terms containing $\dot{\alpha}^2$ so that centrifugal forces applied to the rod are ignored.

We describe the torque Q by the relations [7]

$$Q = \Phi C, \quad L\dot{C}(t) + RC(t) + \Phi\dot{\varphi}(t) = w \tag{1.4}$$

The second of these is the potential balance equation in the motor winding. Here C, L and R are the current, inductance, and ohmic resistance of the coil, respectively, Φ is the magnetic flux, and w is the controlling voltage applied to the motor.

We introduce the new variable

$$u(x,t) = \upsilon(x,t) + x\alpha(t) \tag{1.5}$$

which describes the total deviation of the rod from the OP axis, together with the dimensionless variables u^*, x^*, t^*, w^*

$$u = lu^*, \quad x = lx^*, \quad t = \tau t^*, \quad w = \chi w^* \left(\tau^2 = \frac{\rho l^4}{EI}, \quad \chi = \frac{EIR}{\Phi kL} \right) \tag{1.6}$$

Substituting relations (1.4)–(1.6) into (1.1)–(1.3) and omitting the asterisks, we obtain

$$u''''(x,t) + \ddot{u}(x,t) = 0 \tag{1.7}$$

$$jT_L\dddot{u}'(0,t) + j\ddot{u}'(0,t) + \delta\dot{u}'(0,t) - T_L\dot{u}''(0,t) - u''(0,t) = w \tag{1.8}$$

$$u(0,t) = u''(1,t) = 0, \quad m\ddot{u}(1,t) = u'''(1,t) \tag{1.9}$$

Here j, m, T_L and δ are dimensionless parameters (the moment of inertia of the motor armature, the mass of the load, the electromagnetic time constant and the back EMF coefficient)

$$j = \frac{J}{\rho l^3}, \quad m = \frac{M}{\rho l}, \quad T_L = \frac{L}{R\tau}, \quad \delta = \frac{\Phi^2 k^2}{lR(EI\rho)^{1/2}} \tag{1.10}$$

Relation (1.8), obtained from Eqs (1.2), (1.4) using the equation

$$u'(0,t)=\alpha \tag{1.11}$$

plays the role of a boundary condition for the new boundary-value problem. Equation (1.11) follows from the boundary condition (1.3) $\upsilon'(0, t)=0$.

If we put $T_L=0$ in Eqs (1.7)–(1.9), we obtain equations [3] in which the inductance L of the coil is ignored.

2. STATEMENT OF THE PROBLEM, CONTROL

When $w=0$ the boundary-value problem (1.7)–(1.9) has the solution

$$u(x,t)=Gx \quad (\upsilon(x,t)=0,\ \alpha=G)$$

where G is an arbitrary constant corresponding to an undeformed rod rotated through an angle $\alpha=G$ from the OP axis. When $G=0$ we have

$$u(x,t)=0 \quad (\upsilon(x,t)=0,\ \alpha=0) \tag{2.1}$$

We consider a control in the form of linear feedback which is intended to ensure the asymptotic stability of solution (2.1)

$$T\dot{w}(t)+w(t)=-\beta\dot{u}'(0,t)-\beta_0 u(1,t)-\beta_1\dot{u}(1,t) \tag{2.2}$$

Here $T>0$ is a dimensionless time constant in the control circuit, and β, β_0, β_1 are constant feedback coefficients with respect to the angular velocity $\dot{\alpha}$, the position of the displaced load, and its velocity. To implement the feedback (2.2) it is of course necessary to have appropriate detectors.

The purpose of controlling the manipulator is usually to bring the manipulated object into a required position and to keep it in that position. Hence it is natural to investigate control with position feedback.

Control with feedback of the form (2.2) with $T=0$ has been previously considered [8, 9], and also for feedback with respect to the angle α, its derivative and integral, and with respect to the bending deformation of the rod [3]. Control by sequential displacements of the elastic rod has also been considered [10].

Suppose that with condition (2.1) the manipulated object occupies the required position; we then say that the equilibrium (2.1) is desired. The equilibrium state (2.1) is a solution of the system of equations (1.7)–(1.9) with control (2.2). The linear boundary-value problem (1.7)–(1.9), (2.2) determines an infinite spectrum of eigenvalues λ. Specifying the asymptotic stability problem for the solution of Eq. (2.1), we pose the following problem [3, 10]. In the coefficient space of the feedback (2.2) it is required to construct a region of values for which all the eigenvalues λ satisfy $\operatorname{Re}\lambda<0$.

In addition to (1.7)–(1.9) we consider, for comparison, the equations of motion of a completely rigid arm with a load at its end, controlled by means of the feedback (2.2). In the dimensionless variables given by (1.6) they have the form

$$T_L\left(\frac{1}{3}+m+j\right)\dddot{\alpha}+\left(\frac{1}{3}+m+j\right)\ddot{\alpha}+\delta\dot{\alpha}=w \tag{2.3}$$

$$T\dot{w}+w=-\beta_0\alpha-(\beta+\beta_1)\dot{\alpha} \tag{2.4}$$

The parameters in Eq. (2.3) are described by relations (1.10).

For a completely rigid arm, feedback with respect to the load position, its velocity, and the angular velocity $\dot{\alpha}$ of the arm is equivalent to feedback with respect to the angle α and the angular velocity $\dot{\alpha}$. Hence in expression (2.4) the coefficient of the velocity $\dot{\alpha}$ is the sum $\beta+\beta_1$.

3. THE CHARACTERISTIC EQUATION

We shall seek a solution of the boundary-value problem (1.7)–(1.9), (2.2) in the form

$$u(x,t) = Ke^{\lambda t}X(x)$$

where K is a constant, λ is an eigenvalue, and $X(x)$ is the eigenfunction.

We obtain for the function $X(x)$ the boundary-value problem

$$X''''(x)+\lambda^2 X(x)=0 \tag{3.1}$$

$$[(jT_L\lambda^2+j\lambda+\delta)\lambda X'(0)-(T_L\lambda+1)X''(0)](T\lambda+1)= \tag{3.2}$$

$$=-\beta\lambda X'(0)-(\beta_0+\beta_1\lambda)X(1)$$

$$X(0)=X''(1)=0, \quad m\lambda^2 X(1)=X'''(1) \tag{3.3}$$

We construct a solution to the boundary-value problem (3.1)–(3.3) in the form of the sum

$$X(x)=C_1\sin\nu x+C_2\cos\nu x+C_3\operatorname{sh}\nu x+C_4\operatorname{ch}\nu x \tag{3.4}$$

where $C_1,\ldots,C_4$ are unknown constants and

$$\lambda^2=-\nu^4 \tag{3.5}$$

Substituting expression (3.4) into the boundary conditions (3.2) and (3.3) we obtain a system of linear homogeneous equations for the constants $C_1,\ldots,C_4$. Expanding the determinant of the system, we obtain the characteristic quasipolynomial. The non-zero eigenvalues λ satisfy the following characteristic equation obtained with the help of this quasipolynomial

$$\Delta(\lambda)=R(\lambda)R_1(\nu,m)+\nu(T\lambda+1)(T_L\lambda+1)R_2(\nu,m)+(\beta_0+\beta_1\lambda)R_3(\nu)/\nu=0 \tag{3.6}$$

where

$$R(\lambda)=\lambda[(jT_L\lambda^2+j\lambda+\delta)(T\lambda+1)+\beta], \quad R_1(\nu,m)=Q_1(\nu)+m\nu Q_3(\nu)$$

$$R_2(\nu,m)=Q_3(\nu)+2m\nu Q_2(\nu), \quad R_3(\nu)=\sin\nu+\operatorname{sh}\nu$$

$$Q_1(\nu)=1+\cos\nu\operatorname{ch}\nu, \quad Q_2(\nu)=-\sin\nu\operatorname{sh}\nu, \quad Q_3(\nu)=\operatorname{sh}\nu\cos\nu-\operatorname{ch}\nu\sin\nu$$

$R(\lambda)$ is the characteristic polynomial of the electric motor with feedback (2.2) for $\beta_0+\beta_1=0$; $R_1(\lambda)$ is the characteristic quasipolynomial of the elastic rod with cantilever attachment, and $R_2(\nu, m)$ with hinged attachment.

From Eq. (3.5) we obtain $\lambda=\pm i\nu^2$. Substituting each of these values into (3.6) we obtain two equations in ν. However, the values of λ obtained by solving these equations are the same, because if ν is a root of one of the equations then $i\nu$ is a root of the other. It is thus sufficient to analyse just one of the equations obtained by substituting into (3.6), for example, the expression

$$\lambda=i\nu^2 \tag{3.7}$$

Together with ν Eq. (3.6), (3.7) has the roots $-\nu$, $i\bar{\nu}$, and $-i\bar{\nu}$. The roots $\pm\nu$ and $-i\bar{\nu}$ correspond to the eigenvalues λ and $\bar{\lambda}$.

It follows from Eq. (3.6) that $\Delta(0)=\beta_0$. Suppose r is a real number and $\nu=\nu(r)=r\exp(-i\pi/4)$. Then $\lambda=i\nu^2=r^2$. When $r\to+\infty$ we find from expression (3.6) that $\Delta(\lambda)=\Delta(r^2)\to+\infty$. Hence when $\beta_0<0$ (positive feedback) Eq. (3.6) has a real root $\lambda>0$ and solution (2.1) is unstable for all β, β_1, T, T_L, m, j, δ.

Suppose $\Delta_1(\lambda)=\Delta(\lambda)/\lambda$. Then when $\beta_0<0$ we obtain $\Delta_1(0)=2(\delta+\beta+\beta_1)$. If $\nu=\nu(r)$, then when $r\to+\infty$ we have $\Delta_1(\lambda)=\Delta_1(r^2)\to+\infty$. From this it follows that when $\beta_0=0$, $\delta+\beta+\beta_1<0$ (negative damping) the system is unstable.

To study the asymptotic stability of solution (2.1) we shall use the D-decomposition method [11]. It follows from relation (3.7) that if λ takes imaginary values then ν is imaginary or real. We substitute the value $\lambda=i\varepsilon^2$, into Eq. (3.6), where ε is a real number, and consider its real and imaginary parts

$$\begin{aligned}&-[j(1-TT_L\varepsilon^4)+T\delta]\varepsilon^4R_1(\varepsilon,m)+(1-TT_L\varepsilon^4)\varepsilon R_2(\varepsilon,m)+\beta_0R_3(\varepsilon)/\varepsilon=0\\&[-j(T+T_L)\varepsilon^4+\delta+\beta]R_1(\varepsilon,m)+(T+T_L)\varepsilon R_2(\varepsilon,m)+\beta_1R_3(\varepsilon)/\varepsilon=0\end{aligned}\tag{3.8}$$

Relations (3.8) determine in the system parameter space the image of the real axis $\nu=\varepsilon$, $-\infty<\varepsilon<+\infty$. They do not change when ε is replaced by $-\varepsilon$. Hence the boundary of the asymptotic stability domain in the parameter space can be found by constructing the surface (3.8) for $0\leqslant\varepsilon<\infty$.

4. SYSTEM INSTABILITY WHEN THERE IS NO VISCOSITY

Consider the case

$$\delta=\beta=0\tag{4.1}$$

i.e. when there is no back EMF in the motor and there is no feedback with respect to the angular velocity $\dot{\alpha}$.

For the case (4.1) with

$$T=T_L=m=j=0\tag{4.2}$$

Equation (3.6) takes the form

$$\Delta(\lambda)=\nu(\operatorname{sh}\nu\cos\nu-\operatorname{ch}\nu\sin\nu)+(\beta_0+\beta_1\lambda)(\sin\nu+\operatorname{sh}\nu)/\nu=0,\quad \lambda=i\nu^2\tag{4.3}$$

Let $\lambda_s=i\nu_s^2$ $(s=1,2,\ldots)$ be the roots of Eq. (4.3) with $\beta_0=\beta_1=0$. These are purely imaginary roots. The values of ν_s are given by the asymptotic formula

$$\nu_s\cong\pi/4+\pi s\ \ (s=1,2,\ldots)\tag{4.4}$$

We put $\beta_0=\Delta\beta_0$, $\beta_1=\Delta\beta_1$, $\nu+\nu_s+\Delta\nu_s$, $\lambda+\lambda_s+\Delta\lambda_s$, where $\Delta\lambda_s=2i\nu_s\Delta\nu_s$ and we approximate Eq. (4.3) in a neighbourhood of the point $\beta_0+\beta_1=0$, $\lambda=\lambda_s(\nu=\nu_s)$ by the linear equation

$$-2\nu_s^2\sin\nu_s\operatorname{sh}\nu_s\Delta\nu_s+(\Delta\beta_0+\Delta\beta_1 i\nu_s^2)(\sin\nu_s+\operatorname{sh}\nu_s)=0\tag{4.5}$$

From Eq. (4.5) we obtain

$$\operatorname{Re}\lambda=\operatorname{Re}\Delta\lambda_s=-\frac{\nu_s(\sin\nu_s+\operatorname{sh}\nu_s)}{\sin\nu_s\cdot\operatorname{sh}\nu_s}\Delta\beta_1\tag{4.6}$$

It follows from (4.4) that the quantities (4.6) are positive for "large" odd values of s if $\Delta\beta_1 > 0$, and for "large" even values of s if $\Delta\beta_1 < 0$. Hence if $\beta_1 \neq 0$ and the values of $|\beta_0|$, $|\beta_1|$ are small, then Eq. (4.3) has roots λ such that $\operatorname{Re}\lambda > 0$. From relations (3.8) with conditions (4.1), (4.2) it follows that if $\beta_0\beta_1 \neq 0$, then all roots λ of Eq. (4.3) are such that $\operatorname{Re}\lambda \neq 0$. If $\beta_0 = 0$, then for any $\beta_1 \neq 0$ Eq. (4.3) has the single root $\lambda = 0$, and other roots such that $\operatorname{Re}\lambda \neq 0$. Equation (4.3) only has purely imaginary roots when $\beta_1 = 0$. It follows from Rouché's theorem [12] that each root of (3.6) depends continuously on the parameters β, β_0, β_1, T, T_L, m, j, δ. Hence, when the coefficients β_0 and β_1 are varied, the real part of any root λ of (4.3) can change its sign only when the point (β_0, β_1) intersects the $\beta_1 = 0$ axis. Because for Eq. (4.3) in a small neighbourhood of the point $\beta_0 = \beta_1 = 0$ with $\beta_1 \neq 0$ there are roots λ such that $\operatorname{Re}\lambda > 0$, then from what has been said it follows that there are such roots at any point of the (β_0, β_1) plane when $\beta_1 \neq 0$. Thus, if $\beta_1 \neq 0$, Eq. (4.3) has roots λ such that $\operatorname{Re}\lambda > 0$. Thus in the case (4.1), (4.2) with any coefficients β_0 and $\beta_1 \neq 0$, the solution (2.1) is unstable.

Instead of (4.2) we consider the weaker restriction

$$T = T_L = 0 \tag{4.7}$$

Under the conditions (4.1), (4.7), as under the conditions (4.1), (4.2), Eq. (3.6) has purely imaginary roots only when $\beta_1 = 0$. From this, using the continuity of the dependence of each root of Eq. (3.6) on the parameters β_0, β_1, m, j, we can conclude that the solution (2.1) is also unstable when $\beta_1 \neq 0$ in the more general case (4.1), (4.7).

From Eqs (3.8) we have

$$\beta_0 = [j(1 - TT_L\varepsilon^4) + T\delta]\varepsilon^5 R_1(\varepsilon, m) / R_3(\varepsilon) - (1 - TT_L\varepsilon^4)\varepsilon^2 R_2(\varepsilon, m) / R_3(\varepsilon) \tag{4.8}$$

$$\beta_1 = [j(T + T_L)\varepsilon^4 - \delta - \beta]\varepsilon R_1(\varepsilon, m) / R_3(\varepsilon) - (T + T_L)\varepsilon^2 R_2(\varepsilon, m) / R_3(\varepsilon) \tag{4.9}$$

In the case (4.1) when

$$T_L = 0 \tag{4.10}$$

all the points (4.8), (4.9) $(0 < \varepsilon < \infty)$ on the straight line

$$\beta_1 = T\beta_0 \tag{4.11}$$

If $\beta_0 = 0$, then for all values $\beta_1 \neq 0$ Eq. (3.6) has the single root $\lambda = 0$ and other roots λ such that $\operatorname{Re}\lambda \neq 0$. Equation (3.6) has purely imaginary roots only under condition (4.11).

Consider the three-dimensional parameter space β_0, β_1, T when $0 \leqslant T < \infty$. We recall that if $T = 0$, then there is instability when $\beta_1 \neq T\beta_0$ $(\beta_1 \neq 0)$. From this, using the continuity of the dependence of each root of (3.6) on the parameters β_0, β_1, T, we can conclude that the equilibrium (2.1) is unstable when $\beta_1 \neq T\beta_0$ in the case (4.1), (4.10). Because with condition (4.1) the time constants T and T_L occur symmetrically in relations (4.8) and (4.9), the derived assertions on the instability of the equilibrium (2.1) with load position and velocity feedback remain true if $T = 0$ and $T_L \neq 0$.

Thus load velocity feedback does not compensate for the lack of viscosity and does not stabilize the system.

If for some coefficients β_0 and β_1 the system is unstable in the case (4.1), then it remains unstable for sufficiently small values of the damping coefficients δ and feedback β.

The assertion of the instability of a system with feedback of the form (2.2) for small viscous damping forces proportional to the angular velocity $\dot{\alpha}$ [8, 9] was based on experimental and numerical investigations. The numerical investigations in [9] were performed without taking into account the dynamics of the drive $(j = \delta = T_L = 0)$ and delay in the control circuitry $(T = 0)$. Here this assertion is proved analytically both when the drive dynamics and delay are taken into account and when they are not.

5. STABILITY DOMAINS WHEN THERE IS VISCOSITY

We will denote by β_0^* the smallest local maximum of the function $\beta_0=\beta_0(\varepsilon)$ (4.8) when $0<\varepsilon<\infty$. Under the conditions (4.1), (4.2) $\beta_0^*\cong 10.17$. It can be shown that in the case (4.1), (4.10) when $\beta_0>\beta_0^*$, $\beta_1=T\beta_0$, Eq. (3.8) has roots λ such that $\operatorname{Re}\lambda>0$, and the system is therefore unstable. For $0<\beta_0<\beta_0^*$, $\beta_1=T\beta_0$ the system is stable, but not asymptotically so. The same is also true if $\delta=\beta=T=0$, $T_L\neq 0$. When $T_L=0$, $\delta>0$, $\beta>0$ there exists in the β_0, β_1 plane a domain of asymptotic stability (DAS). If $\delta\to 0$, $\beta\to 0$ this domain contracts to the interval $0<\beta_0<\beta_0^*$, $\beta_1=T\beta_0$. When $T=0$, $\delta>0$, $\beta>0$ a DAS exists "around" the interval $0<\beta_0<\beta_0^*$, $\beta_1=T_L\beta_0$ only when $j=0$. If however $j\neq 0$, then there is no asymptotic stability outside this interval.

Under conditions (4.7) one can introduce the variable $\beta_1/(\delta+\beta)$ instead of β_1 in relation (4.9). Figure 2 shows the domains of asymptotic stability for the variables β_0, $\beta_1/(\delta+\beta)$ according to formula (4.8), (4.9) when $T=T_L=j=0$ and $m=0, 1, 2$. (These domains lie inside the loops and are bounded on the left by the $\beta_0=0$ axis). The DAS for a completely rigid body (2.3) with control (2.4) under conditions (4.7) is unbounded and is described by the inequalities

$$\beta_0>0,\quad \beta_1>-\delta-\beta \tag{5.1}$$

It follows from known results [3] that the domains of asymptotic stability for a pliable arm with feedback with respect to the angle α (with coefficient β_0) and angular velocity $\dot\alpha$ (with coefficient $\beta+\beta_1$) in the case (4.7) are also described by inequalities (5.1). The domains of asymptotic stability shown in Fig. 2 are bounded and are completely contained in domain (5.1).

When there are no viscous forces proportional to the angular velocity $\dot\alpha$, load velocity feedback has no stabilizing effect on the system (see Section 4). Consideration of Fig. 2 shows, however, that when $\delta+\beta\neq 0$, as β_1 increases up to a certain value the range of stable values of β_0 increases. For example, if $m=1$, then when $\beta_1=0$ only the interval $0<\beta_0<3.2$ is asymptotically stable, while when $\beta_1=10$ the interval $0<\beta_0<25$ is asymptotically stable. The bigger the mass m, the larger the range of stability for β_0 when $\beta_1\neq 0$. If $m\to\infty$, then, as can be shown using relations (4.8) and (4.9), the stability domain "fills" the entire corner $\beta_0>0$, $\beta_1>(\beta_0/3-1)(\delta+\beta)$. Thus, for sufficiently large values of m all the points within this corner lie in the stability domain, and all the points outside this corner are outside the stability domain. Hence load velocity feedback can stabilize the system when there is no viscosity.

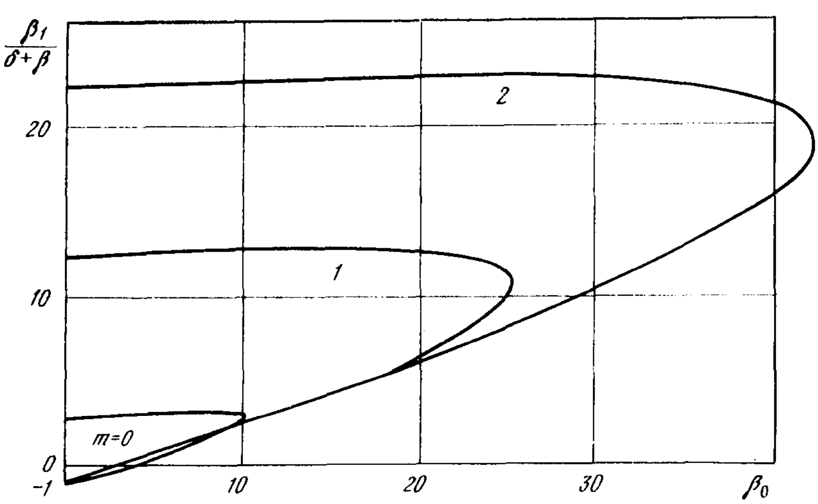

FIG. 2.

In Fig. 3 formulae (4.8) and (4.9) are used to construct the DAS when $T_L = j = \delta = 0$, i.e. without taking into account the motor dynamics, for $T = 0$, 1, $m = 2$, $\beta = 1$, 2, 3. When β decreases the DAS decreases in size, and when $\beta \to 0$ it contracts to an interval on the straight line (4.11) and vanishes, as stated above. The DAS of an absolutely rigid body (2.3) with control (2.4) when $T_L = 0$ is unbounded and is given by the inequalities

$$\beta_0 > 0, \quad \beta_1 > -\delta - \beta + \frac{T(1/3 + m + j)}{T\delta + 1/3 + m + j}\beta_0 \tag{5.2}$$

It follows from known results [3] that the DAS for a pliable arm with feedback with respect to the angle α and angular velocity $\dot{\alpha}$ with $T_L = 0$ is also unbounded. The domains shown in Fig. 3 (situated inside the loops to the right of the $\beta_0 = 0$ axis) are bounded. Each of them is entirely contained in the corresponding domain (5.2), and this can be shown analytically using (4.8) and (4.9).

The reduction of the DAS when the rod pliability is considered emphasizes the importance of this consideration when investigating control processes.

Comparison with results obtained previously [3] shows that in the cases considered with feedback with respect to the angle α and angular velocity $\dot{\alpha}$ the DAS in the feedback coefficient space is larger than for load position and velocity feedback.

6. STABILITY DOMAINS WHEN $\beta_1 = 0$

If $\beta_1 = 0$, then from Eqs (3.8) we have an expression for β_0 in accordance with (4.8), and, moreover

$$\beta = -\delta + j(T + T_L)\varepsilon^4 - \varepsilon(T + T_L)R_2(\varepsilon, m) / R_1(\varepsilon, m) \tag{6.1}$$

We denote by $\varepsilon_i = \varepsilon_i(m)$ $(i = 1, 2, \ldots)$ the zeros of the quasipolynomial $R_1(\varepsilon, m)$.

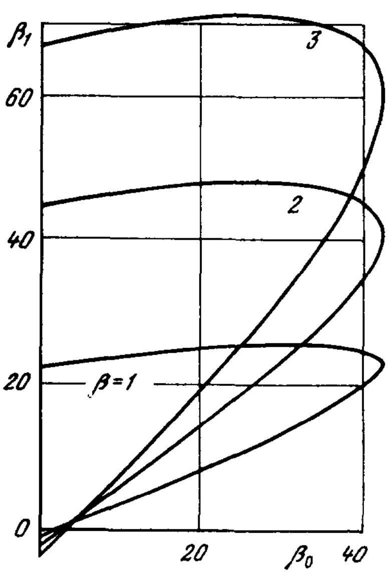

FIG. 3.

By considering relations (3.6), (3.8), (4.8) and (6.1) it can be shown that with condition (4.7) the DAS occupies the "half-strip"

$$\beta_0(\varepsilon_1) > \beta_0 > 0, \quad \beta > -\delta \tag{6.2}$$

where the function $\beta_0(\varepsilon)$ is defined by (4.8).

If $T \neq 0$ or (and) $T_L \neq 0$, then the parametric equations (4.8) and (6.1) describe an infinite number of branches. The first branch V_1 is obtained for $0 < \varepsilon < \varepsilon_1$. It begins when $\varepsilon = 0$ at the point

$$\beta_0 = 0, \quad \beta = -\delta \tag{6.3}$$

When $\varepsilon \to \varepsilon_1 - 0$ we have $\beta_0(\varepsilon) \to \beta_0(\varepsilon_1)$, $\beta(\varepsilon) \to +\infty$ and the branch V_1 tends to the asymptote A_1. Each of the subsequent branches V_i $(i = 2, 3, \ldots)$ is obtained when $\varepsilon_{i-1} < \varepsilon < \varepsilon_i$. On each of them the quantity β increases strictly monotonically from $-\infty$ to $+\infty$ as ε increases. Hence none of the branches intersects itself and each of them divides the β_0, β_1 plane into two parts. The branch V_i $(i \geqslant 2)$ has two asymptotes A_{i-1} and A_i: it approaches the first when $\varepsilon \to \varepsilon_{i-1} + 0$ and the second when $\varepsilon \to \varepsilon_i - 0$. The asymptotes are parallel to the β axis and are described by the equations $\beta_0 = \beta_0(\varepsilon_i)$. All branches V_i $(i = 1, 2, \ldots)$ intersect the β axis. If $T = 0$ and $T_L \neq 0$, the intersections occur at the point (6.3). If $T \neq 0$ and $T_L = 0$, all intersections except the first lie below the point (6.3) and as $i \to \infty$ they move away from it strictly monotonically and tend to $-\infty$. If however $T \neq 0$ and $T_L \neq 0$, the points of intersection of the branches V_i for $2 \leqslant i \leqslant p$ lie below the point (6.3), and for $p + 2 \leqslant i$ they lie above the point

$$\beta_0 = 0, \quad \beta = \delta T / T_L \tag{6.4}$$

The number p is equal to the maximum value of i for which $1 - TT_L\varepsilon_i^4 > 0$. The branch V_{p+1}, and only that branch, intersects the β axis twice: the first time below the point (6.3), and the second time above the point (6.4). As i increases those points of intersection which lie below the point (6.3) move away from it strictly monotonically, while those above the point (6.4) approach the latter strictly monotonically and tend to it as $i \to \infty$.

We will take the positive direction of motion along each branch to be the direction for which the parameter ε, and hence β, increases. Here each of the branches divides the β_0, β_1 plane into a left part and a right part. To construct the stability and instability domains it is important to establish whether the number of eigenvalues λ for which $\operatorname{Re}\lambda > 0$ increases or decreases as the point β_0, β crosses each branch from left to right [11]. Analysis of Eqs (3.6), (4.8), (6.1) shows that the number of eigenvalues λ for which $\operatorname{Re}\lambda > 0$ increases on crossing from left to right the branch on which $R_1(\varepsilon, m) > 0$, and decreases on crossing the branch on which $R_1(\varepsilon, m) < 0$.

Bearing in mind the properties of the branches V_i $(i = 1, 2, \ldots)$ described above, we can show the following. Let $T = 0$ and $T_L \neq 0$. If we also have $j \neq 0$, then there are no values of β_0 and β_1 for which the system is asymptotically stable. If however $j = 0$, then a DAS exists, bounded on the left by the β axis and on the right by the curve formed by the V_i branches. The DAS for $T = j = m = 0$, $T_L = 0.02$, $\delta = 2$ is constructed in Fig. 4. Its right-hand boundary is the branch V_1, tending to the asymptote A_1. This domain lies in the half-strip (6.2). Let $T_L = 0$ and $T \neq 0$. If we also have $\delta \neq 0$ or $\delta = j = 0$, then a DAS exists and is similar to the domain shown in Fig. 4. It too is an unbounded set. If however $\delta = 0$ but $j \neq 0$, the system is not asymptotically stable for any values of β_0 and β_1.

It can be shown that if $T \neq 0$ but $T_L \neq 0$, then the DAS is bounded, and its left boundary is a section of the β axis satisfying the conditions

$$-\delta \leqslant \beta \leqslant \delta T / T_L \tag{6.5}$$

The right boundary is constructed using the branches V_i. Its ends are the points (6.3), (6.4)

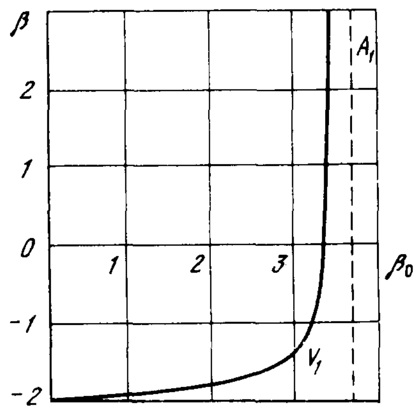

FIG. 4.

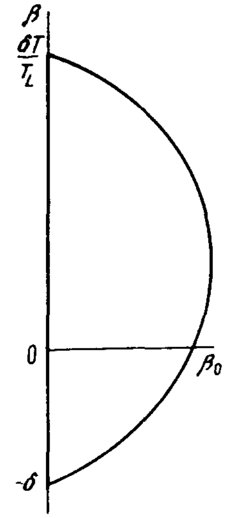

FIG. 5.

(Fig. 5). In the β_0, β plane this domain lies in the strip (6.5).

We note the following fact which can be strictly proved by analysing relations (3.8) in the complete parameter space β, β_0, β_1. If $T\delta - T_L\beta > 0$ and $T_L > 0$, $j > 0$, then when $\beta_1 \neq 0$ there is no asymptotic stability. This in the certain sense supplements the stability picture.

In the above, in a number of cases where one or another parameter tends to zero, the DAS does not shrink to zero continuously, but disappears "instantly" ("in a jump"), or "instantly" appears. This apparently paradoxical loss of continuity can occur in systems with infinite spectra and does not occur in systems with a finite number of degrees of freedom.

REFERENCES

1. LUR'YE A. I., *Analytical Mechanics*. Fizmatgiz, Moscow, 1961.
2. VOL'MIR A. S., *Stability of Deformable Systems*. Nauka, Moscow, 1967.
3. TIMOSHENKO S. P., *Oscillations in Engineering*. Nauka, Moscow, 1967.
4. LAVROVSKII E. K. and FORMAL'SKII A. M., On the stabilization of the angular position of an elastic rod. *Izv. Akad. Nauk SSSR. Tekhn. Kibern.*, **6**, 115–123, 1989.
5. AKULENKO L. D. and BOLOTNIK N. N., On the controlled rotation of an elastic rod. *Prikl. Mat. Mekh.* **46**, 6, 587–595, 1982.
6. BERBYUK V. Ye. and DEMIDYUK M. V., On the controlled motion of an elastic manipulator with distributed parameters. *Izv. Akad. Nauk SSSR. Mekh. Tverd. Tela*, **2**, 58–67, 1984.
7. CHILIKIN M. G. and SANDLER A. S., *A General Course on Electric Drives*. Energoizdat, Moscow, 1981.
8. CANNON R. H. Jr and SCHMITZ E., Initial experiments on the end-point control of a flexible one-link robot. *Int. J. Robot. Res.* **3**, 3, 62–75, 1984.
9. CETINKUNT S. and YU W.-L., Closed-loop behaviour of a feedback-controlled flexible arm: a comparative study. *Int. J. Robot. Res.* **10**, 3, 263–275, 1991.
10. LAVROVSKII E. K. and FORMAL'SKII A. M., Stabilization of a specified position of an elastic rod. *Prikl. Mat. Mekh.* **53**, 5, 752–760, 1989.
11. NEIMARK Yu. I., *Dynamical Systems and Controlled Processes*. Nauka, Moscow, 1978.
12. SHABAT B. V., *Introduction to Complex Analysis*, Pt. 1. Nauka, Moscow, 1976.

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